ON THE INCREMENTS OF PLASTIC DEFORMATIONS AND THE YIELD SURFACE

(O PRIRASHCHENII PLASTICHESKOI DEFORMATSII I POVERKHNOSTI TEKUCHESTI)

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If to a body in equilibrium under the action of a system of forces Q_M an additional system of forces ΔQ_M is applied gradually at some instance and then removed, then along such closed (in regard to the variation of the system of forces) loading path the work of the system ΔQ_M must be positive or zero. This is Drucker's [1] postulate of the theory of plasticity. The significance and the vary validity of this postulate is limited to materials which exhibit stable strain hardening for all possible forms of loading. The converse is also true, i.e. if some of the states are unstable then the work of ΔQ_M for such states is negative. Such states are encountered for some materials even under simple tension. As an example the "plasticity kink" can be mentioned, (the upper bound of elasticity range), observed for mild steel. One may expect the appearance of such states in polycrystalline materials under a complex system of loadings. Consider, e.g. an aggregate of grains which is deformed in various directions. Such an aggregate may be unstable in regard to very small shearing displacements, which change sharply their directions.

Drucker's postulate, for our purpose, is valid for all possible plastic deformations of material bodies, with exception perhaps of some singular points of the deformation path. In what follows we shall ignore these exceptional points. However, the coincidence of the plastic-deformation increment vector with the normal to the yield surface, and the convexity of the yield surface do not follow automatically from this postulate for a general case. It is necessary to stress these facts since they have been accepted so far as theoretically proved principles. But as we will see, this is correct only under the assumption that the plastic deformation of a body subjected to an arbitrary loading is not accompanied by a noticable variation of the elastic properties of the material or, since the question of the deformational anisotropy, which appears during the process of plastic flow, is one of the cardinal questions, it is essential to obtain for a general case the strict theoretical conclusions following from Drucker's postulate.

We shall employ vector notation used in [2]. Let the stress vector of a homogeneously deformable body in the stress space σ describe a path L, which is determined by a function $\sigma(t)$. At an instant $t = t_k$ let $\sigma_k = \sigma(t_k)$. We shall call the path L together with σ_k a point k. By the unloading from the point k and then elastically reloading along various directions we can obtain points T of the yield surface $f_k(\sigma_T) = 0$. Inside of this surface the plastic deformations remain constant, and outside of it appear additional plastic deformations. In connection with Drucker's postulate consider an arbitrary fixed point M determined by a vector σ_M inside of f_k, a point T determined by vector σ_T on f_k and a point P in the neighborhood of T outside of f_k determined by vector σ_p. Consider next a closed loading-unloading path $MTPTM$. The process along the paths MT and PTM (without ternary passing beyond the elastic limit) is conservative provided that the elastic limit was not exceeded. This means that these processes do not depend upon the trajectories but only on the points (M, T) and (P, M). This is because the corresponding deformations are elastic. In order that all states along this path should be defined, the path TP itself must be well defined. The same applies to path L for point k. Let path TP be a segment of a straight line determined by a unit vector $\mathbf{e}$, and let the magnitude of TP be s. For simplicity path MT will also be taken as a segment of a straight line determined by a unit vector $\mathbf{e}_1$ and having magnitude s_1. Consequently,

$$
\begin{aligned}
&(MT) \qquad \boldsymbol{\sigma} - \boldsymbol{\sigma}_M = \Delta\boldsymbol{\sigma} = s_1'\mathbf{e}_1, \qquad (0 \leqslant s_1' \leqslant s_1) \\
&(TP) \qquad \boldsymbol{\sigma} - \boldsymbol{\sigma} \ \ = \Delta\boldsymbol{\sigma} = s_1\mathbf{e}_1 + s'e, \qquad (0 \leqslant s' \leqslant s)
\end{aligned}
\tag{1}
$$

The hypothesis of linear elasticity consists in that the elastic part of the total deformation satisfies

$$Э = Э^e + Э^p \tag{2}$$

i.e. $Э^e$ is a linear, homogeneous function of the strain

$$Э^e = (\varepsilon)\,\boldsymbol{\sigma} \tag{3}$$

where the symmetric matrix (ϵ) of the elastic constants depends on the previous plastic deformations. It is constant inside of f_k and is determined by point k, it is also constant during the reversed motion along the path PTM and is determined by point P, (here point P is understood to represent point k, path TP and σ_p); along the path TP, however, (ϵ) varies from $(\epsilon)_T$ to $(\epsilon)_p$ and it is a continuous function of s'.

Note that the hypothesis of linear elasticity is not satisfied exactly, and thus (3) should be viewed as an approximation. From this hypothesis it follows that

$$\begin{array}{lll} (MT) & d\vartheta = d\vartheta^e = (\varepsilon)_k\, d\sigma = (\varepsilon)_T d\sigma, & d\sigma = \mathbf{e}_1\, ds_1{}' \\ (TP) & d\vartheta = d\vartheta^p + (\varepsilon)_s\, d\sigma + \frac{d}{ds'}(\varepsilon)\,\sigma\, ds', & d\sigma = \mathbf{e}\, ds' \\ (PT) & d\vartheta = d\vartheta^e = (\varepsilon)_p\, d\sigma, & d\sigma = \mathbf{e}\, ds' \ldots \\ (TM) & d\vartheta = d\vartheta^e = (\varepsilon)_p\, d\sigma, & d\sigma = \mathbf{e}_1\, ds_1{}' \end{array} \tag{4}$$

Drucker's postulate asserts that the work is non-negative

$$W = \int\limits_{MTPTM} \Delta\sigma d\vartheta \tag{5}$$

In calculating the work W we omit small quantities of second and higher orders in s and of third and higher orders in s_1. Thus in the expansions

$$(\varepsilon)_{s'} = (\varepsilon)_T + s'\left[\frac{d}{ds}(\varepsilon)\right]_T + \cdots \qquad (\varepsilon)_p = (\varepsilon)_T + s\left[\frac{d}{ds}(\varepsilon)\right]_T + \cdots \tag{6}$$
$$\frac{d}{ds'}(\varepsilon)_{s'} = \left[\frac{d}{ds'}(\varepsilon)\right]_T + s'\left[\frac{d^2}{ds^2}(\varepsilon)\right]_T + \cdots$$

it is sufficient to retain terms which appear above. Obviously we have

$$\int\limits_{PTM} \Delta\sigma\, d\vartheta^e = -\int\limits_{MTP} \Delta\sigma\, d\vartheta^e = -\int\limits_{MTP} \Delta\sigma\,(\varepsilon)_p\, d\sigma$$

and therefore

$$W - \int\limits_{TP} \Delta\sigma d\vartheta^p = \int\limits_{MT} \Delta\sigma\,[(\varepsilon)_T - (\varepsilon)_P]\, d\sigma +$$
$$+ \int\limits_{TP} \Delta\sigma\,[(\varepsilon)_{s'} - (\varepsilon)_P]\, d\sigma + \int\limits_{TP} \Delta\sigma\, \frac{d}{ds'}(\varepsilon)\,\sigma\, ds'$$

Taking into account (1), (4) and (6), we conclude that the first integral on the right-hand side of (b) is a small quantity of the order $ss_1{}^2$, the second of the order s^2s_1, and the third of the order ss_1. The integral which appears on the left-hand side can be represented, with an accuracy to the order of s^2s_1 as follows:

$$\int\limits_{TP} \Delta\sigma\, d\vartheta^p = s_1 s \mathbf{e}_1 \left(\frac{d\vartheta^p}{ds}\right)_T$$

Retaining small quantities of the order ss_1 and $ss_1{}^2$, and dropping s^2s_1, ..., results in

$$W = ss_1\mathbf{e}_1\left[\left(\frac{d\vartheta^p}{ds}\right)_T + (\varepsilon)_T{}'\,\sigma_M\right] + \frac{1}{2}\, ss_1{}^2\mathbf{e}_1\,(\varepsilon)_T{}'\,\mathbf{e}_1 \tag{7}$$

The derivatives

$$\frac{d\mathfrak{э}^p}{ds}, \qquad (\varepsilon)_T{}' = \frac{d}{ds}(\varepsilon)$$

are computed at the point T along the path TP.

The vector

$$Э' = \left[\frac{d\mathfrak{э}^p}{ds} + (\varepsilon)'\,\boldsymbol{\sigma}\right]_T \tag{8}$$

is independent of the position of point M, i.e. of the direction of the vector $\mathbf{e}_1$. Let now point M be moved to the point T_1 located on the surface f_k at a distance s_1 from T. Approaching to the limit with $s \to 0$, $s_1 \to 0$ under the condition that $s/s_1 \to 0$, and taking into account that vector $\mathbf{e}_1$ in the limit is tangent to the surface f_k at the point T, we obtain from the Drucker's postulate

$$\mathbf{e}_1 \cdot Э' \geqslant 0$$

for an arbitrary $\mathbf{e}_1$ lying in the tangential plane. This is possible only in the case when the vector $Э'$ is directed along the normal $\mathbf{n}$ to the surface f_k at the point T:

$$Э' = |Э'| \cdot \mathbf{n} \tag{9}$$

Consequently, from the positiveness of the work increment, it follows that the form of the stress and plastic deformation relationship is

$$\frac{d\mathfrak{э}^p}{ds} + (\varepsilon)'\,\boldsymbol{\sigma} = G\,\mathrm{grad}\, f \tag{10}$$

Where $f(\boldsymbol{\sigma}) = 0$ is the equation of the yield surface, $ds = |d\boldsymbol{\sigma}|$, and $(\epsilon)'$ is a derivative of the elastic-constants matrix (inverse quantities of the moduli) with respect to s' at a point $\boldsymbol{\sigma}$. The expression (10) differs from a commonly-used form because of the presence of a second term $(\epsilon)\boldsymbol{\sigma}$. In order to account correctly for the elastic properties of materials, the evaluation of the magnitude of this term should be made in comparison with the magnitude $(\epsilon)\, d\sigma/ds$. The ratio of these magnitudes even with respect to the modulus

$$\delta = \left|\frac{d(\varepsilon)\,\boldsymbol{\sigma}}{(\varepsilon)\,d\boldsymbol{\sigma}}\right| \tag{11}$$

is not small *a priori*. For instance, for a simple tension ($\epsilon = 1/E$, E - elasticity modulus)

$$\delta = -\frac{\sigma dE}{E d\sigma}$$

will be of the order of one for a number of materials. Thus, the principle of the coincidence of the plastic-deformation increment vector with the normal to the yield surface is true in general only under certain restrictive assumptions in regard to the influence of the elastic properties of the medium. Substantial deviations from this principle should be expected in the neighborhood of the corner points along the deformation and stress paths, where the influence of the elastic properties is particularly strong.

Let us now construct a two-dimensional plane Π containing vector $\mathbf{e}_1$ and a normal $\mathbf{n}$ at point T. This plane will intersect the yield surface $f_k = 0$ along some plane curve, and a tangential hypersurface along a straight line $T \sim x$ containing point T. The distance z from this straight line to the curve measured in the plane Π along the direction parallel to the exterior normal $\mathbf{n}$ at a point x, within the accuracy of small quantities of higher order, is expressed by the curvature κ, which is considered to be positive if the curve is concave

$$z = \frac{1}{2} \varkappa x^2$$

The distance of a point T_1 of the surface which coincides with the point M to the point T is approximately s_1, and the elevation of point T, from the curve $T - x$ (for $\kappa > 0$) is $1/2\ \kappa s_1{}^2$. Thus, to the second order of approximation, vector $\mathbf{e}_1$ (directed from T_1 to T) forms with the normal $\mathbf{n}$ an obtuse angle $(1/2)\pi + (1/2)\kappa s_1$, and $\mathbf{e}_1\mathbf{n}_1 = 1/2\ \kappa s_1$.

The work W in (7), after taking into account that $\sigma_M = \sigma_T - s_1\mathbf{e}_1$, to the second order of approximation, according to (9) can be represented as

$$W = \frac{1}{2} s s_1{}^2 [\mathbf{e}_1'(\varepsilon)_T'\mathbf{e}_1 - \varkappa\,|\,\mathrm{Э}'\,|] \qquad (12)$$

From the condition $W > 0$ it follows

$$\varkappa\,|\,\mathrm{Э}'\,| < \mathbf{e}_1\,(\varepsilon)_T'\mathbf{e}_1 \qquad (13)$$

Here $\mathbf{e}_1$ is a unit vector of an arbitrary tangent line to the surface f_k at T, and $(\epsilon)_T' = d(\epsilon)/ds$ is a symmetric matrix whose elements are the derivatives of (ϵ) with respect to an arbitrary unit vector $\mathbf{e}$, which is directed either outside of f_k or is tangent to it. Clearly, under such circumstances we have no reason to assert that a symmetrical quadratic form of the directional cosines

$$\mathbf{e}_1\ (\varepsilon)_T'\,\mathbf{e}_1 = \varepsilon_{ij}'\,\gamma_i\gamma_j \qquad (i, j = 1, 2, \ldots, 5) \qquad (14)$$

will be positive for arbitrary k, $\mathbf{e}_1$ and $\mathbf{e}$. Indeed, taking into account that the elastic constants ϵ_{ij} form a matrix which is an inverse of the elastic moduli matrix E_{ij}, and that in a number of known simplest cases they increase with increasing plastic deformation (the moduli decrease), there exist certainly such materials and such loading paths for which the form (14) is negative. Moreover, according to (13) the curvature κ can be either negative (convex) or positive (concave). (If for simple loading the moduli are decreasing, then it is most probable that for reversed plastic deformations they are restored to a certain degree of their initial values, i.e. $\mathbf{e}_1(\epsilon)'\mathbf{e}_1 < 0$).

Thus, the hypothesis of the convexity of f_k likewise has no theoretical basis, unless we neglect elastic properties of materials. For an ideally rigid-plastic body this hypothesis, as well as the hypothesis of the coincidence discussed above, is certainly theoretically well grounded.

BIBLIOGRAPHY

1. Drucker, D.C., Variational principles in the mathematical theory of plasticity. Calculus of variations and its applications. *Proc. of Symposia in Applied Mathematics*, Vol. 8, 1958.

2. Il'iushin, A.A., O sviazi mezhdu napriazheniiami i malymi deformatsiami v mekhanike vkliuchennykh sred (On the connection between the stresses and small deformations in the mechanics of inclusions). *PMM* Vol. 18, No. 6, 1954.

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